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Implicit Functions, Nonlinear Integral Equations, and the Measure of Noncompactness of the Superposition Operator

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INTRODUCTION

The most important integral operators in nonlinear functional analysis are the Urysohn operator

$$Ux(t) = \int_0^1 k(t, s, x(s))\, ds \tag{1}$$

and the Hammerstein operator

$$Hx(t) = \int_0^1 k(t, s) f(s, x(s))\, ds. \tag{2}$$

The latter one is usually written as product $H = KF$ of the nonlinear superposition operator

$$Fx(s) = f(s, x(s)) \tag{3}$$

and the linear integral operator

$$Ky(t) = \int_0^1 k(t, s)\, y(s)\, ds. \tag{4}$$

Therefore, in the classical theory (see e.g. [8, 17]) one is interested in sufficient conditions for the compactness of K and continuity of F in certain function spaces to prove the existence of a solution of the Hammerstein equation $x - Hx = o$ by means of the Schauder fixed point principle. From the viewpoint of applications, however, it seems worthwhile to extend the theory to a larger class of mappings. Thus, if we set $\|K\|_\infty :=$

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$\inf\{\|K+C\| \mid C \text{ linear, compact}\}$ [5, 10] then it may happen that for a *noncompact* integral operator K

$$0 < \|K\|_\infty < \|K\|. \tag{5}$$

In this case, if we choose any function f such that

$$\|K\|_\infty < 1/\chi(F) < \|K\|, \tag{6}$$

where $\chi(F)$ denotes the usual Hausdorff measure of noncompactness of the nonlinear operator F (see below), then the operator H is strictly condensing but neither compact nor contracting.

Further, it may occur that one (trivial) solution of a nonlinear problem is known a priori, but one is interested in other (nontrivial) solutions. A typical way of treating such problems is to consider equations involving a parameter and to apply some kind of implicit function theorem (see e.g. [18]). In order, to be of use one must replace the continuous differentiability by the weaker notion of "linearization" ([19]).

The purpose of this paper is twofold. In the first section we will calculate explicitly the measure of noncompactness $\chi(F)$ of the superposition operator F in the spaces C and L_p in order to determine all functions f satisfying (6) for a given K. In the second part we will combine the notion of "linearization" and "asymptotic linearization" of a nonlinear operator with the theory of condensing maps to get abstract implicit function theorems as well as existence theorems for integral equations of Hammerstein type.

Before going into Section 1 let us recall the exact definition of $\chi(F)$. Given a bounded subset M of a normed space X, the *Hausdorff measure of noncompactness* $\chi(M)$ is defined as the infimum of all positive ε such that there exists a finite ε-net for M in X. A nonlinear operator $D\colon X \to X$ is called χ-bounded if

$$\chi(D) := \inf\{k \mid k > 0,\ \chi(D(M)) \leqslant k\chi(M) \text{ for all } M \subseteq X\} < \infty,$$

and strictly χ-condensing if $\chi(D) < 1$ [14].

In Section 2 we will need the following result on fixed points of condensing operators:

PROPOSITION 1. *Let $D\colon X \to X$ be continuous and strictly condensing, and suppose that*

$$D(S_R) \subseteq T_R \tag{7}$$

for some $R > 0$. (Here and in the sequel by $T_R[S_R]$ we denote the closed ball [sphere] with center o and radius R). *Then D has a fixed point in T_R.*

Proof. The proof follows easily from Theorem 1 in [12].

COMPUTATION OF $\chi(F)$

There is a large number of papers on sufficient conditions for the complete continuity of the Urysohn operator (1) (for continuous functions, e.g., in [9] and [15], for Hölder continuous functions, e.g., in [11], for summable functions, e.g., in [16]) and continuity of the superposition operator (3) (for Hölder continuous functions, e.g., in [2] and [3], for summable functions, e.g., in [6] and [13]). However, it is a well known fact that the operator F in general fails to be *completely continuous*, which means nothing else than $\chi(F) > 0$.

Surprisingly enough, nobody has tried to calculate its exact measure of noncompactness, as far as we know. It turns out that a necessary and sufficient condition for the operator F to be χ-bounded is that the function f satisfies an appropriate Lipschitz condition.

THEOREM 1. *Let f: $J \times \mathbb{R} \to \mathbb{R}$ be continuous and consider* (3) *as operator from the space $C(J)$ (with the usual norm $\|x\|_\infty = \max_{t\in J} |x(t)|$, $J := [0, 1]$) into itself. Then the following three conditions are equivalent*:

(i) $|f(t, u) - f(t, v)| \leqslant k\,|u - v|$ *for all* $t \in J$ *and all* $u, v \in \mathbb{R}$,

(ii) $\|Fx - Fy\|_\infty \leqslant k\,\|x - y\|_\infty$ *for all* $x, y \in C(J)$,

(iii) $\chi(F(M)) \leqslant k\chi(M)$ *for all bounded* $M \subset C(J)$.

Proof. Condition (i) implies (ii) trivially. If (ii) holds and $\{z_1,..., z_m\}$ is a finite ε-net for M in $C(J)$, then obviously $\{Fz_1,..., Fz_m\}$ is a finite $k\varepsilon$-net for $F(M)$ in $C(J)$ and therefore (iii) holds.

Assume now that (i) is false; i.e., there exist $t_0 \in J$ and $u_0, v_0 \in \mathbb{R}$ such that

$$|f(t_0, u_0) - f(t_0, v_0)| > k\,|u_0 - v_0|; \tag{8}$$

without loss of generality assume $0 < t_0 < 1$ and $u_0 < v_0$. Now, let M_0 be the set of all functions x_n ($n = 1, 2, 3,...$) defined by

$$\begin{aligned} x_n(t) &:= v_0, && 0 \leqslant t \leqslant t_0,\\ &:= (v_0 - u_0)[1 - (t - t_0)/\delta_n] + u_0, && t_0 < t < t_0 + \delta_n,\\ &:= u_0, && t_0 + \delta_n \leqslant t \leqslant 1, \end{aligned}$$

($\delta_n := (1 - t_0)/n$). Then clearly

$$\chi(M_0) \leqslant \tfrac{1}{2}(v_0 - u_0) \tag{9}$$

since the distance of all functions x_n from the single function $z_1(t) \equiv \frac{1}{2}(u_0 + v_0)$ is less than or equal to the right-hand side of (9). We claim that

$$\chi(F(M_0)) \geqslant \tfrac{1}{2}\,|f(t_0, u_0) - f(t_0, v_0)| =: \gamma_0. \tag{10}$$

In fact, let us assume that there is a finite ε_0-net $\{z_1,..., z_m\}$ for $F(M_0)$ in $C(J)$, where $\varepsilon_0 < \gamma_0$. Then for each $n \in \mathbb{N}$ there exists $j \in \{1,..., m\}$ with $|f(t, x_n(t)) - z_j(t)| \leqslant \varepsilon_0$ $(t \in J)$. Since z_j and f are continuous functions, we have $|z_j(t_0 + \delta_n) - z_j(t_0)| \leqslant \frac{1}{2}(\gamma_0 - \varepsilon_0)$ and $|f(t_0 + \delta_n, u_0) - f(t_0, u_0)| \leqslant \frac{1}{2}(\gamma_0 - \varepsilon_0)$ for sufficiently large n, and hence

$$\begin{aligned}
&|f(t_0, u_0) - f(t_0, v_0)| \\
&\quad\leqslant |f(t_0, u_0) - f(t_0 + \delta_n, u_0)| + |f(t_0 + \delta_n, u_0) - z_j(t_0 + \delta_n)| \\
&\qquad + |z_j(t_0 + \delta_n) - z_j(t_0)| + |z_j(t_0) - f(t_0, v_0)| \\
&\quad\leqslant \gamma_0 + \varepsilon_0 < 2\gamma_0
\end{aligned}$$

contradicting (10). Now the assertion follows from (8), (9) and (10).

Remark 1. The proof that (ii) implies (iii) heavily relies on the fact that the operator F is defined on the whole space $C(J)$, because otherwise the functions Fz_j need not be defined. In fact, there are operators satisfying a Lipschitz condition of type (ii) but not being χ-bounded (but being bounded, of course, with respect to the *Kuratowski* measure of noncompactness).

Conversely, there are simple operators which are χ-bounded in the sense of (iii) but do not satisfy any Lipschitz condition, e.g.

$$F(\xi_1, \xi_2, \xi_3,...) = (\sqrt{|\xi_1|}, k\xi_2, k\xi_3,...)$$

in the space m of bounded real sequences with the sup norm.

Remark 2. Let $C^\alpha(J)$ $(0 < \alpha \leqslant 1)$ be the space of all Hölder continuous functions, equipped with the norm $\|x\|_\alpha = \max\{\|x\|_\infty, h_\alpha(x)\}$, where $h_\alpha(x) := \sup_{s \neq t} |x(s) - x(t)|/|s - t|^\alpha$. If we consider (3) as operator from the space $C^\alpha(J)$ into itself, then condition (i) of the preceding theorem is not sufficient for F to be Lipschitz, as the following example shows:

Let $f(t, u) = |u|$, $x(t) = \delta - t$ $(0 < \delta < 2^{1/\alpha})$, and $y(t) = -t$. Then $\|x - y\|_\infty = \delta$, $h_\alpha(x - y) = 0$, $\|Fx - Fy\|_\infty = \delta$, and $h_\alpha(Fx - Fy) = 2\delta^{1-\alpha}$. Therefore (ii) can not be satisfied since $\|Fx - Fy\|_\alpha / \|x - y\|_\alpha = 2\delta^{-\alpha} \to \infty$ as $\delta \to 0$.

One sufficient condition for F to be Lipschitz in $C^\alpha(J)$ is of course that

$$|f(t, u) - f(t, v) - f(s, x) + f(s, y)| \leqslant k\,|u - v - x + y|$$

holds for all s, $t \in J$ and all u, v, x, $y \in \mathbb{R}$.

The proof of an analogous result for L_p spaces requires a little more technique since we have to consider functions on the whole interval J. For continuous f the proof of the equivalence between (i) and (ii) is due to Berkolajko [1]. Here we assume only a Carathéodory condition for f (i.e., $f(t, \cdot)$ is continuous for almost all $t \in J$ and $f(\cdot, u)$ is measurable for all $u \in \mathbb{R}$).

THEOREM 2. *Let $f: J \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and consider (3) as operator from the space $L_p(J)$ (with the usual norm $\|x\|_p = [\int_0^1 |x(t)|^p\, dt]^{1/p}$, $1 \leqslant p < \infty$) into itself. Then the following three conditions are equivalent*:

(i) $|f(t, u) - f(t, v)| \leqslant k\, |u - v|$ *for almost all* $t \in J$ *and all* $u, v \in \mathbb{R}$,

(ii) $\|Fx - Fy\|_p \leqslant k\, \|x - y\|_p$ *for all* $x, y \in L_p(J)$,

(iii) $\chi(F(M)) \leqslant k\chi(M)$ *for all bounded* $M \subset L_p(J)$.

Proof. Condition (i) implies (ii) and (ii) implies (iii) as above. Conversely, if (ii) holds, then for every $D \subseteq J$ with positive measure we have also

$$\int_D |f(t, x(t)) - f(t, y(t))|^p\, dt \leqslant k^p \int_D |x(t) - y(t)|^p\, dt.$$

Dividing by mes D and letting mes D tend to zero, one obtains (i) (see also Theorem 15 in [4]).

Now assume that (ii) is false; i.e., there are functions $x_0, y_0 \in L_p(J)$ such that

$$\|Fx_0 - Fy_0\|_p > k\, \|x_0 - y_0\|_p. \tag{11}$$

Let $\{t_0, t_1, ..., t_n\}$ be a partition of J which is so fine that x_0 and y_0 are piecewise constant (without loss of generality) on $[t_{i-1}, t_i)$, say,

$$x_0(t) \equiv u_i, \qquad y_0(t) \equiv v_i \qquad (t_{i-1} \leqslant t < t_i;\ i = 1, ..., n).$$

Let M_0 be the set of all functions $z \in L_p(J)$ taking either the value u_i or the value v_i on $[t_{i-1}, t_i)$. Then again

$$\chi(M_0) \leqslant \tfrac{1}{2}\, \|x_0 - y_0\|_p, \tag{12}$$

since the distance of all functions in M_0 from the single function $z_1(t) \equiv \frac{1}{2}(u_i + v_i)$ $(t_{i-1} \leqslant t < t_i)$ is less than or equal to the right-hand side of (12). We claim again that

$$\chi(F(M_0)) \geqslant \tfrac{1}{2}\, \|Fx_0 - Fy_0\|_p. \tag{13}$$

In fact, given any finite ε-net $\{z_1, ..., z_m\}$ for $F(M_0)$ in $L_p(J)$ we can *assume* again without loss of generality that z_j is piecewise constant, say

$$z_j(t) \equiv c_{ij} \qquad (t_{i-1} \leqslant t < t_i;\ i = 1, ..., n;\ j = 1, ..., m).$$

Now, if we define

$$\begin{aligned} z_0(t) &:= f(t, u_i), \qquad t_{i-1} \leqslant t < \tau_i, \\ &:= f(t, v_i), \qquad \tau_i \leqslant t < t_i, \end{aligned}$$

$(\tau_i = \frac{1}{2}(t_{i-1} + t_i);\ i = 1, ..., n)$, we can also assume that

$$f(t, u_i) \equiv a_i \qquad (t_{i-1} \leqslant t < \tau_i), \qquad f(t, v_i) \equiv b_i \qquad (\tau_i \leqslant t < t_i).$$

Therefore we have

$$\begin{aligned} \varepsilon^p &\geqslant \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |z_o(t) - z_j(t)|^p\, dt \\ &= \sum_{i=1}^{n} \int_{t_{i-1}}^{\tau_i} |a_i - c_{ij}|^p\, dt + \int_{\tau_i}^{t_i} |b_i - c_{ij}|^p\, dt \\ &= \sum_{i=1}^{n} \tfrac{1}{2}(t_i - t_{i-1})[|a_i - c_{ij}|^p + |b_i - c_{ij}|^p] \\ &\geqslant \sum_{i=1}^{n} \tfrac{1}{2}(t_i - t_{i-1})\, 2^{-p+1}\, |a_i - b_i|^p = 2^{-p}\, \|Fx_0 - Fy_0\|_p^p, \end{aligned}$$

and (13) is proved.

Let us now consider the case where the operator F acts from one space $L_p(J)$ into another space $L_q(J)$ $(1 \leqslant q < p < \infty)$. In this case the proof of the equivalence between (ii) and (iii) is essentially the same (with $\|Fx - Fy\|_p$ in (ii) replaced by $\|Fx - Fy\|_q$). Since we can consider the constant k in (i) as a L_∞ function $k = k(t)$, one should expect that (ii) is equivalent to

$$|f(t, u) - f(t, v)| \leqslant k(t)\, |u - v| \qquad \text{for all } u, v \in \mathbb{R}, \tag{14}$$

where $k \in L_{pq/(p-q)}(J)$. Indeed, condition (14) is sufficient for (ii). However, the following example shows that for the converse implication one must proceed more carefully, because there exist functions f which can not be estimated linearly in the second argument, but generate operators F of linear growth. This somewhat surprising fact makes it likely that Theorem 2 is false for different p and q.

EXAMPLE 1 (for $p = 2$, $q = 1$). Let $f: J \times \mathbb{R} \to \mathbb{R}$ be defined by

$$f(t, u) := \frac{u^2}{t}\left(1 - \log \frac{u^2}{t}\right), \qquad u^2 \leqslant t,$$

$$:= 1, \qquad u^2 > t.$$

Then there is *no* function $k \in L_2(J)$ such that

$$|f(t, u)| \leqslant k(t)\,|u| \qquad (u \in \mathbb{R}). \tag{15}$$

Indeed, the function $f(t, u)/u = u^{-1}$ is monotonically decreasing in u for $u^2 > t$, and hence has maximum $1/\sqrt{t}$ which is not square integrable. On the other hand, the corresponding superposition operator $F: L_2(J) \to L_1(J)$ *does* satisfy a condition of the form

$$\|Fx\|_1 \leqslant k\,\|x\|_2 \qquad (x \in L_2(J)) \tag{16}$$

as one can see as follows: A simple computation shows that

$$f(t, u) = \inf_{0 < \epsilon < \infty} \left[e^{-\epsilon t} + \varepsilon u^2\right]$$

and hence

$$f(t, x(t)) \leqslant e^{-\epsilon t} + \varepsilon x(t)^2$$

and

$$\|Fx\|_1 \leqslant \|a_\epsilon\|_1 + \varepsilon\,\|x\|_2^2 \leqslant 1/\varepsilon + \varepsilon\,\|x\|_2^2$$

where $a_\epsilon(t) := e^{-\epsilon t}$. But the last expression in this inequality (as a function of ε) attains its minimum for $\varepsilon = 1/\|x\|_2$, and therefore (16) holds with $k = 2$.

Another example of the same type is provided by the function

$$f(t, u) := u^2\sqrt{2^n}, \qquad u^2 \leqslant 1, \quad 2^{-n} \leqslant t < 2^{-n+1},$$

$$:= u^2, \qquad u^2 > 1, \quad 2^{-n} \leqslant t < 2^{-n+1}.$$

Let us return to Theorem 2. It turns out that an analogous theorem holds for different p and q if we replace condition (i) by a whole family of Lipschitz conditions involving a positive real parameter ε:

THEOREM 3. *Let $f: J \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and consider (3) as operator from the space $L_p(J)$ into the space $L_q(J)$ $(p > q)$. Then the following three conditions are equivalent:*

(i) *for all* $\varepsilon > 0$ *there exists a function* $a_\varepsilon \in L_q(J)$ *such that* $\|a_\varepsilon\|_q \leqslant c\varepsilon^{-q/(p-q)}$ *and* $|f(t,u) - f(t,v)| \leqslant a_\varepsilon(t) + \varepsilon\,|u-v|^{p/q}$ *for all* $u, v \in \mathbb{R}$.

(ii) $\|Fx - Fy\|_q \leqslant k\,\|x-y\|_p$ *for all* $x, y \in L_p(J)$,

(iii) $\chi(F(M)) \leqslant k\chi(M)$ *for all bounded* $M \subset L_p(J)$.

Proof. The proof of the equivalence between (ii) and (iii) remains unchanged. Now, if (i) holds with some fixed $\varepsilon > 0$, then

$$\begin{aligned}\|Fx - Fy\|_q^q &\leqslant \|a_\varepsilon\|_q^q + \varepsilon^q\,\|x-y\|_p^p \\ &\leqslant c^q\varepsilon^{-q^2/(p-q)} + \varepsilon^q\,\|x-y\|_p^p.\end{aligned}$$

In particular, the special choice $\varepsilon := c^{(p-q)/p}[q/(p-q)]^{(p-q)/pq}\,\|x-y\|_p^{-(p-q)/q}$ yields

$$\|Fx - Fy\|_q^q \leqslant c^{q(p-q)/p}[q/(p-q)]^{-q/p}[p/(p-q)]\,\|x-y\|_p^q$$

which is (ii) with $k = k(c; p, q) = c^{(p-q)/p}\,[q/(p-q)]^{-1/p}[p/(p-q)]^{1/q}$.

Conversely, if (ii) holds, set, for $\varepsilon > 0$

$$\varphi_\varepsilon(t,u,v) := \max\{|f(t,u) - f(t,v)| - \varepsilon\,|u-v|^{p/q}, 0\}$$

and $J_+ := \{t \mid t \in J,\ |f(t,u) - f(t,v)| > \varepsilon\,|u-v|^{p/q}\}$. For $t \in J_+$ we have $|\varphi_\varepsilon(t,u,v)|^q \leqslant |f(t,u) - f(t,v)|^q - \varepsilon^q\,|u-v|^p$. Now fix $x, y \in L_p(J)$. If we denote the $L_p(J_+)$ norm of $x-y$ by η and if $n := [\eta^p k^{-r}\varepsilon^r]$ (where $r := pq/(p-q)$ and $[\vartheta]$ denotes as usual the entire part of ϑ) we can divide J_+ into subsets $J_1, \ldots, J_{n+1}$ such that

$$\int_{J_i} |x(t) - y(t)|^p\,dt \leqslant k^r\varepsilon^{-r} \qquad (i = 1, \ldots, n+1).$$

From (ii) it follows that

$$\int_{J_i} |f(t,x(t)) - f(t,y(t))|^q\,dt \leqslant k^r\varepsilon^{-qr/p} \qquad (i = 1, \ldots, n+1).$$

Hence by the estimation

$$\begin{aligned}&\int_J |\varphi_\varepsilon(t,x(t),y(t))|^q\,dt \\ &\quad\leqslant \sum_{i=1}^{n+1}\int_{J_i} |f(t,x(t)) - f(t,y(t))|^q\,dt - \varepsilon^q\sum_{i=1}^{n+1}\int_{J_i}|x(t)-y(t)|^p\,dt \\ &\quad\leqslant (n+1)\,k^r\varepsilon^{-qr/p} - \varepsilon^q\eta^p \leqslant (n+1)\,k^r\varepsilon^{-qr/p} - nk^r\varepsilon^{-qr/p} \\ &\quad= k^r\varepsilon^{-qr/p} = k^{pq/(p-q)}\varepsilon^{-q^2/(p-q)},\end{aligned}$$

we have shown that the function $\psi_\epsilon(t) := \varphi_\epsilon(t, x(t), y(t))$ is in $L_q(J)$ for all $x, y \in L_p(J)$. By a standard argument (see [7, p. 37]) the function

$$a_\epsilon(t) := \sup_{u, v \in \mathbb{R}} \varphi_\epsilon(t, u, v)$$

is also in $L_q(J)$ and $\|a_\epsilon\|_q \leqslant c^{-q/(p-q)}$ with $c = c(k; p, q) = k^{p/(p-q)}$. Since the relation

$$|f(t, u) - f(t, v)| \leqslant a_\epsilon(t) + \varepsilon\, |u - v|^{p/q}$$

is obvious, Theorem 3 is proved.

Implicit Functions and Hammerstein Equations

As we are now able to estimate the measure of noncompactness of both the nonlinear operator (3) and the linear operator (4), we want to formulate theorems on the solvability of Hammerstein integral equations in terms of fixed point theorems for condensing operators. Moreover, since we consider Hammerstein equations involving a parameter λ, we will apply some implicit function theorem.

Definition 1. ([19]) Let X, Y be Banach spaces, Λ a metric space with metric ρ, $x_0 \in X$, $\lambda_0 \in \Lambda$, and let Ω and ω be neighborhoods of x_0 in X and λ_0 in Λ, respectively. Let $A\colon \omega \times \Omega \to Y$ be a nonlinear operator and $A(\lambda_0, x_0) = o$. If there exist both an operator $\hat{A}\colon \omega \to L(X, Y)$ and a function $x_0\colon \omega \to X$ with $\lim_{\lambda \to \lambda_0} x_0(\lambda) = x_0$ such that

$$\lim_{\lambda \to \lambda_0} \inf_{0 < \rho \leqslant r} \sup_{\|x - x_0(\lambda)\| = \rho} (1/\rho)\, \|A(\lambda, x) - \hat{A}(\lambda)(x - x_0(\lambda))\| = 0 \tag{17}$$

for some sufficiently small $r > 0$, $\hat{A}(\lambda)$ is called a *linearization of the operator A along the function x_0*.

Definition 2. In the above notation, let $B\colon \omega \times X \to Y$ be a nonlinear operator (the x_0 is not needed here). If there exists an operator $\hat{B}(\cdot\,; \infty)$: $\omega \to L(X, Y)$ such that

$$\lim_{\lambda \to \lambda_0} \inf_{\rho \geqslant r} \sup_{\|x\| = \rho} (1/\rho)\, \|B(\lambda, x) - \hat{B}(\lambda; \infty)\, x\| = 0 \tag{18}$$

for some sufficiently large $r > 0$, then $\hat{B}(\lambda; \infty)$ is called an *asymptotic linearization of the operator B*.

For example, if A is (Fréchet) differentiable in x_0 with derivative A', then $\hat{A}(\lambda) \equiv A'$, and if B is asymptotically linear with asymptotic derivative $B'(\infty)$ ([7]), then $\hat{B}(\lambda; \infty) \equiv B'(\infty)$. On the other hand, the scalar function $A(\lambda, x) = a(\lambda)\, x^\alpha$ is not differentiable in $x_0 = 0$ if $\alpha < 1$, but has linearization $\hat{A}(\lambda) \equiv O$ provided $\lim_{\lambda \to \lambda_0} a(\lambda) = 0$; similarly, the function $B(\lambda, x) = b(\lambda)\, x^\beta$ is not asymptotically linear if $\beta > 1$, but has asymptotic linearization $\hat{B}(\lambda; \infty) \equiv O$ provided $\lim_{\lambda \to \lambda_0} b(\lambda) = 0$. Moreover, the limit condition on a [on b] implies that the operator $A(\lambda, \cdot)$ [the operator $B(\lambda, \cdot)$] leaves all balls of small radius [of large radius] invariant for λ close to λ_0; this is not true if, for example, $a(\lambda) = b(\lambda) \equiv 1$.

In general, the following "principle on invariant balls" holds for operators with asymptotic linearization (for the usual linearization see [19]):

LEMMA 1. *Let the operator B admit an asymptotic linearization $\hat{B}(\lambda; \infty)$ and suppose that there exist $\delta > 0$ and $M > 0$ such that*

$$\|\hat{B}(\lambda; \infty)^{-1}\| \leqslant M \tag{19}$$

for $\rho(\lambda, \lambda_0) < \delta$. Let $D(\lambda)\colon X \to X$ be defined by

$$D(\lambda)\, x := x - \hat{B}(\lambda; \infty)^{-1}\, B(\lambda, x).$$

Then there exists a radius $R \geqslant r$ (not depending on λ) such that

$$D(\lambda)\, S_R \subseteq T_R$$

for $\rho(\lambda, \lambda_0) < \delta$.

Proof. The proof is almost obvious: By (18) we can choose $\delta > 0$ such that

$$\inf_{\rho \geqslant r} \sup_{\|x\| = \rho} (1/\rho)\, \|B(\lambda, x) - \hat{B}(\lambda; \infty)\, x\| < 1/M$$

for $\rho(\lambda, \lambda_0) < \delta$. Hence for some $R \geqslant r$ we have

$$\begin{aligned} \sup_{\|x\| = R} \|D(\lambda)\, x\| &\leqslant \|\hat{B}(\lambda; \infty)^{-1}\| \sup_{\|x\| = R} \|\hat{B}(\lambda; \infty)\, x - B(\lambda, x)\| \\ &\leqslant MR(1/M) = R. \end{aligned}$$

Combining Lemma 1 and Proposition 1 from the introduction we obtain the following implicit function theorem:

THEOREM 4. ***Suppose that $B\colon \omega \times X \to Y$ is continuous and admits an asymptotic linearization $\hat{B}(\lambda; \infty)$ satisfying (19) for $\lambda \in \omega$. Let $\phi(\lambda)\colon X \to X$ be defined by***

$$\phi(\lambda)\, x := B(\lambda, x) - \hat{B}(\lambda; \infty)\, x$$

and assume

$$\chi(\phi(\lambda)\,Q) \leqslant q(\delta, r)\,\chi(Q) \qquad (\rho(\lambda, \lambda_0) < \delta,\ Q \subseteq T_r), \tag{20}$$

where $q(\delta, r) \to 0$ as $\delta \to 0$. Then for small $\rho(\lambda, \lambda_0)$ there exists at least one solution $x = x(\lambda) \in T_R$ of the equation $B(\lambda, x) = o$.

Proof. Obviously, the zeros of the operator $B(\lambda, \cdot)$ coincide with the fixed points of the operators $D(\lambda)$. Moreover, since

$$\chi(D(\lambda)) \leqslant M\chi(\phi(\lambda)) \leqslant Mq(\delta, r),$$

$D(\lambda)$ is strictly condensing for λ sufficiently close to λ_0. Therefore the statements of Lemma 1 and Proposition 1 yield the existence of fixed points $x(\lambda)$ of $D(\lambda)$ for these λ.

Let us now indicate how to apply Theorem 4 to the Hammerstein equation $x = Hx$. Therefore, setting

$$B(\lambda, x)(t) := \int_0^1 k(t, s)\, f(s, x(s); \lambda)\, ds - x(t) \tag{21}$$

we are interested in answering the following three questions:

(a) Which condition yields the existence of an asymptotic linearization $\hat{B}(\lambda; \infty)$ of (21)?

(b) When does condition (19) hold for $\rho(\lambda, \lambda_0) < \delta$?

(c) Which condition on f implies (20)?

Let us make some observations. First of all, the linearization of the operator B and the superposition operator

$$F(\lambda, x)(t) = f(t, x(t); \lambda), \tag{22}$$

respectively, are related by the formula

$$\hat{B}(\lambda; \infty) = I - K\hat{F}(\lambda; \infty)$$

and hence $\hat{B}(\lambda; \infty)$ is invertible, if the norm of $K\hat{F}(\lambda; \infty)$ is sufficiently small. Moreover, the measure of noncompactness of the operator $\phi(\lambda)$ can be estimated by the measure of noncompactness of the operator

$$\Psi(\lambda)\, x := F(\lambda, x) - \hat{F}(\lambda; \infty)\, x,$$

namely,

$$\chi(\phi(\lambda)) = \chi(K\Psi(\lambda)) \leqslant \|K\|_\infty\, \chi(\Psi(\lambda)).$$

Thus the above questions can be reformulated more precisely as follows:

(a′) Which condition yields the existence of an asymptotic linearization $\hat{F}(\lambda;\infty)$ of (22)?

(b′) When does the condition $\|\hat{F}(\lambda;\infty)\| < 1/\|K\|$ hold for $\rho(\lambda,\lambda_0) < \delta$?

(c′) Which condition on f implies the χ-boundedness of $F(\lambda,\cdot)$ (and hence of $\Psi(\lambda)$) for $\rho(\lambda,\lambda_0) < \delta$?

Concerning (a′) we can state the following rather obvious criterion:

LEMMA 2. *If the limit*

$$g(t;\lambda) := \lim_{u\to\infty} (1/u) f(t,u;\lambda) \tag{23}$$

exists and if $G(\lambda,x)(t) := g(t;\lambda)\,x(t)$ *satisfies the relation*

$$\lim_{\lambda\to\lambda_0} \inf_{\rho\geqslant r} \sup_{\|x\|=\rho} (1/\rho)\,\|F(\lambda,x) - G(\lambda,x)\| = 0,$$

then

$$\hat{F}(\lambda;\infty)\,x = G(\lambda,x). \tag{24}$$

Proof. The proof follows directly from the definition of the asymptotic linearization.

Let us mention that (23) is not necessary for F to have an asymptotic linearization. The function $f(t,u;\lambda) = u \sin u$, for example, does not satisfy (23), but the corresponding operator F has asymptotic linearization $\hat{F}(\lambda;\infty) \equiv O$.

Under the assumptions of Lemma 2 it is also possible to answer (b′), because the norm $\|\hat{F}(\lambda;\infty)\|$ can be estimated via the norms $\max_{t\in J} |g(t;\lambda)|$ or $\operatorname{vrai\,max}_{t\in J} |g(t;\lambda)|$, if F acts in the space $C(J)$ or $L_p(J)$, respectively. Finally, (c′) was already answered in the first section, and so we are done.

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